Introduction

Let $G = (V, E)$ be a graph. Suppose there is a transmitter located at each vertex $v \in V$ capable of broadcasting at strength $0, 1, \ldots, k$, where strength 0 corresponds with not broadcasting. A vertex $v$ broadcasting at strength $s$ is heard by all vertices within distance $s$ of $v$.

Our goal is to assign strengths to the transmitters such that every vertex not transmitting hears the broadcast by one that is. The result is a dominating $k$-limited broadcast on $G$. The cost of such a broadcast is the sum of the strengths of the transmitters. $\gamma_{k,l}(G)$ denotes the least cost of a $k$-limited broadcast on $G$. Observe that $\gamma_{k,1} = \gamma$.

Limited broadcast domination is a restriction of broadcast domination (where vertices can broadcast at any strength) introduced in [ErWo91].

Broadcast domination is known on grid graphs [BS09]; however, $k$-limited broadcast on grid graphs is unknown. We provide tight bounds on the 2-limited broadcast domination number of grid graphs.

2-Limited Broadcasts Domination

Let $G = (V, E)$ be a graph. For each vertex $i \in V$, let $x_{i,k} = \begin{cases} 1 & \text{if vertex } i \text{ is broadcasting at strength } k, \\ 0 & \text{otherwise} \end{cases}$

Formulation of $\gamma_{2,2}(G)$ as an Integer Linear Program (ILP):

Minimize: $\sum_{k=1}^{2} \sum_{i \in V} k \cdot x_{i,k}$

Subject to: $\sum_{d(i,j) \leq k} x_{i,k} \geq 1$, for each vertex $j \in V$

Example 1: Red diamonds represent vertices at their center broadcasting at strength 1 or 2.

[Image of optimal 2-limited broadcast on $P_5 \bigcirc P_5$, $\gamma_{2,2}(P_5 \bigcirc P_5) = 29$.]

Upper Bounds

For $P_m \bigcirc P_n$, where $2 \leq m \leq 12$, we create upper bounds using the following methodology.

Methodology:
1. Fix $m$ and use an ILP solver [CO17] to determine $\gamma_{2,2}(P_m \bigcirc P_n)$ for small values of $n$ $(\leq 50)$.
2. Manually inspect for patterns in the broadcast structure.
3. Create general constructions based on these patterns.

Example 2: Referring to Example 1, we observe a pattern in the optimal 2-limited broadcast on $P_5 \bigcirc P_5$. Using this pattern, we repeatedly tile $P_5 \bigcirc P_5$ with a main tile $B$ and complete the ends of the broadcast with $B_1$, $B_2$, or $B_3$ based on $n \pmod{4}$.

n $\equiv$ 0 (mod 4): $B + B + \cdots + B + B_1$, 

n $\equiv$ 1 (mod 4): $B + \cdots + B + B_2$,

n $\equiv$ 2 (mod 4): $B_1 + B + \cdots + B + B_3$,

n $\equiv$ 3 (mod 4): $B + \cdots + B + B_1$.

Resulting upper bound: $\gamma_{2,2}(P_5 \bigcirc P_5) \leq n + 1$.

Through a counting argument on the number of broadcasting vertices in the plane, we establish our generalized upper bound.

Fractional 2-limited Multipacking

The dual of the linear programming relaxation of 2-limited broadcast domination is 2-limited multipacking $mp_2(G)$, stated as follows.

Let $G = (V, E)$ be a graph. For each vertex $i \in V$, let weight packed at vertex $i = y_i \in [0, 1]$.

Formulation of $mp_2(G)$ as a Linear Program (LP):

Maximize: $\sum_{i \in V} y_i$

Subject to (1): $\sum_{d(i,j) \leq 1} y_i \leq 1$, for each vertex $j \in V$,

(2): $\sum_{d(i,j) \leq 2} y_i \leq 2$, for each vertex $j \in V$.

Example 3: Red circles depict vertices at their center packed with weight 0.5.

Lower Bounds

For $P_{m=12} \bigcirc P_n$, we create lower bounds by the following methodology.

Methodology:
1. Fix $m$, given the main tile $P_m \bigcirc P_n$ used in our 2-limited broadcast construction on $P_{m} \bigcirc P_n$, use an LP solver to determine $mp_2(P_m \bigcirc C_n)$.
2. We can repeatedly tile $P_m \bigcirc P_n$ with this optimal 2-limited multipacking on $P_m \bigcirc C_n$ and create a lower bound on $\gamma_{2,2}(P_m \bigcirc P_n)$.

For $P_{m=12} \bigcirc P_{n=13}$, we use a similar argument as our upper bound and consider the optimal 2-limited multipacking on the plane.

Results

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<th>Lower Bounds</th>
<th>Upper Bounds</th>
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Take Home: We established tight bounds and known optimal values for $\gamma_{2,2}(P_m \bigcirc P_n)$. Using similar methods, we have also created tight bounds and known optimal values for $\gamma_{2,2}(P_m \bigcirc C_n)$ and $\gamma_{2,2}(C_n \bigcirc C_m)$.

References

[BS09] Bodlaand Brook and Simon Spacapan, Broadcast domination of products of graphs, Ars Combinatoria 92 (2009).


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