Berkovich and Uncu's Conjectures on Integer Partitions

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Introduction and Purpose

- An integer partition of n is a way of writing n as an unordered sum of positive integers.
- For example, partitions of 4 are 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$.
- The study of those integer partitions for which parts come from a specified interval is a very active area of research.
- The purpose of this work is to prove four conjectures of Berkovich and Uncu regarding relative sizes of two closely related sets consisting of integer partitions whose parts come from a specified interval.

Motivation

Berkovich and Uncu proved some intriguing results regarding the relative sizes of certain sets of integer partitions. For any $L \geq 2$,

- $A_{L,1}$ denotes the set of partitions where the smallest part is 1, all parts are $\leq L+1$, and L is not a part (i.e. L is *impermissible*);
- $A_{L,2}$ denotes the set of partitions with parts in the set $\{2, 3, \dots, L+1\}$.

They proved that for any N, the number of partitions of N in $A_{L,1}$ is more than the number of those in $A_{L,2}$.

- Berkovich and Uncu observed that the above result may be viewed as the initial stage of a more general conjecture.

Generalization of Sets

- $C_{L,s,1}$ denotes the set of partitions where the smallest part is s, all parts are $\leq L+s$ and $L+s-1$ is impermissible;
- $C_{L,s,2}$ denotes the set of partitions with parts in the set $\{s+1, \dots, L+s\}$.

Conjecture for general s

Conjecture ⋆ For given positive integers $L \geq 3$ and s, there exists an M, which only depends on s, such that for every $N \geq M$, the number of partitions of N in $C_{L,s,1}$ is more than the number of those in $C_{L,s,2}$.

- Berkovich and Uncu asked similar questions about the q-series analogue of the above conjecture. However, this time they considered a general impermissible part k instead of $L+s-1$.

Conjecture for general k

- The *q-Pochhammer symbol* is defined as

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

- The series $H_{L,s,k}(q)$ is defined as

$$H_{L,s,k}(q) = \frac{q^s(1-q^k)}{(q^s;q)_{L+1}} - \left(\frac{1}{(q^{s+1};q)_L} - 1\right).$$

Conjecture ⋆⋆ For $k \geq s+1$, $H_{L,s,k}(q)$ is eventually positive.

Comparisons

- Conjecture ⋆⋆ is stronger than conjecture ⋆ in the sense that it deals with a general impermissible part k, instead of dealing only with $k = L+s-1$.
- However, Conjecture ⋆⋆ relaxes the condition on the bound. It can depend on all of L, s and k, instead of depending only on s.

Our Main Result

For positive integers L, s and k, with $L \geq 3$ and $k \geq s+1$, the coefficient of q^N in $H_{L,s,k}(q)$ is positive whenever $N \geq \Gamma(s)$, where $\Gamma(s)$ can be written explicitly in terms of s only.

Strengths of our Result

- This result is stronger than the above conjectures because:
  (a) It deals with a general impermissible part k and at the same time produces a bound which depends only on s.
  (b) Moreover, the bound is explicitly known.

Methods and Techniques

- Our proofs involve constructing injective maps between the relevant sets of integer partitions.
- To construct these maps, we make very frequent use of concepts from elementary number theory, especially Frobenius numbers, which are given as follows.
- For natural numbers a and b such that $\gcd(a,b) = 1$, the Frobenius number of a and b is the smallest number n_0 such that the equation $ax + by = n$ has a nonnegative integer solution (x_n, y_n) for all $n \geq n_0$. Sylvester proved that the Frobenius number of a and b is $(a-1)(b-1)$.
- We also frequently use some simple consequences of the division algorithm in the definitions of our maps.

Proofs of Zang and Zeng

- Zang and Zeng also gave proofs of the above conjectures.
- However, while their methods are somewhat more straightforward than ours, they produce results that are asymptotic and therefore do not give explicit bounds.
- In contrast, our methods are combinatorial, and we produce explicit bounds on when $H_{L,s,k}(q)$ has positive coefficients.

Fourth Conjecture

- Berkovich and Uncu defined the series

$$G_{L,2}(q) = \sum_{\substack{s(\pi)=2,\\ l(\pi)-s(\pi)\leq L}} q^{|\pi|} - \sum_{\substack{s(\pi)\geq 3,\\ l(\pi)-s(\pi)\leq L}} q^{|\pi|},$$

where $s(\pi)$ and $l(\pi)$ denote the smallest and largest parts of π, respectively.

- They conjectured that

$$G_{L,2}(q) + q^3 \succeq 0 \quad \text{for} \quad L \geq 5,$$
$$G_{4,2}(q) + q^3 + q^9 \succeq 0,$$
$$G_{3,2}(q) + q^3 + q^9 + q^{15} \succeq 0.$$

- We also proved this conjecture.

References

References

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