Pathwise integration over rough paths for model-free finance
Jose Aviléz 1

1Department of Statistics and Actuarial Science, University of Waterloo

The mathematical finance formalism

In the standard formalism of mathematical finance, the price of a risky asset over a time interval \([0, T]\) is modeled by a semimartingale \(Y = (Y_t)_{t \in [0,T]}\) on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\). At this point, the undergraduate student may feel discouraged by the appearance of high-tech probabilistic machinery to discuss financial questions. The learned reader might ask: what if I picked the wrong semimartingale?

Pedagogically, one might angle towards a probability-free formulation of finance. (Ω,\[0\leq t \leq T\],\(\mathcal{F}_t\)) is modelled by a semimartingale \((Y_t)\) depending on choice of \(\pi\).

Applications and extensions of pathwise Itô calculus

- Black-Scholes and pathwise hedging of payoffs with local volatility [1]
- Pathwise Tanaka formula [7]
- Hitting and pricing formulas for exotics [10, 4]
- Functional pathwise Itô calculus [2, 5]
- Pathwise Itô formula for paths with arbitrary regularity [3]

Uniqueness of quadratic variation

Since the expression \(QV\) depends on the choice of partition \(\pi\), it is profitable to find conditions under which \(QV\) is uniquely determined. First, we define a sequence of balanced partitions \(\pi\):

\[
QV = \lim_{n \to \infty} \lim_{k \to \infty} \sum_{s \in \pi} \mathbb{E}[Y_{s+1} - Y_s] \mathbb{1}_{[0,T]}(s)
\]

where there exists a \(c > 0\) such that for all \(n \geq 1\) we have

\[
\frac{\mathbb{E}[Y_{n+1} - Y_n]}{\mathbb{E}[Y_n]} \leq c
\]

Theorem (Cont & Das, 2017). Let \(\pi\) and \(\tau\) be balanced partition sequences with vanishing mesh \(Y \in \mathcal{C}([0,T])\) for some \(0 < a < b < c\). Suppose \(Y \in \mathcal{A}([0,T] \cap (\mathcal{F}_t)_{t \in [0,T]}\) with \(Y_{s+1} - Y_s\) strictly increasing under \(\pi\) and \(\tau\). Then, under certain regularity and roughness conditions, for all \(a \in [0, T]\): \(QV_{\pi}\) is unique.

Research question: Can this theorem be extended for arbitrary \(\pi\) variation?

Do asset prices admit quadratic variation?

Gatheral, Jaisson, and Rosenbaum [8] recently observed that empirical paths of daily realised volatility time series are closer to Brownian motion.

Pathwise models for (fractional) Brownian paths

It is useful to study classes of continuous functions whose paths have unusual features of (fractional) Brownian motion: (i) nowhere differentiable, (ii) modulus of continuity, (iii) pathwise \(\pi\) variation.

Taking-

Landsberg functions

Define the Faber-Schauder basis \(e_n := (\max((1 - t)^n), \min((1 + t)^n))\) n \to \infty. Set:

\[
Y_H = \sum_{n=1}^{\infty} \sum_{a \in C_n} \mathbb{E}[Y_{a+1} - Y_a] \mathbb{1}_{[0,T]}(a)
\]

where \(H \in \mathcal{C}([0,T])\) is called the Takagi-Landsberg function with Hurst index \(H\).

The Takagi-Landsberg function has several exciting properties: it is nowhere differentiable, it has strictly increasing \(\pi\) variation, it shares characteristics with Brownian bridges.

Research question: What properties of the Takagi-Landsberg function make it amenable to analysis under a rough Itô pathwise calculus?

References